微積分

主題一：參數式與極坐標（Ⅰ）

重點提示

1. 參數微分法

$x = x(t) \qquad y = y(t)$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)}$

$\dfrac{d^2y}{dx^2} = \dfrac{x'(t)y''(t) - y'(t)x''(t)}{\left[x'(t)\right]^3}$

2. 切線斜率公式

$C: \begin{cases} x = x(t) \\ y = y(t) \end{cases}$ 在 $t = t_0$ 處之切線斜率 $m = \left.\dfrac{y'(t)}{x'(t)}\right|_{t=t_0}$

$C: y = f(x)$ 在 $x = x_0$ 處之切線斜率 $m = f'(x_0)$

$C: r = f(\theta)$ 在 $\theta = \theta_0$ 處之切線斜率 $m = \left.\dfrac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}\right|_{\theta=\theta_0}$

3. 弧長公式

$C: \begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad t \in [a,b], \quad L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt$

$C: y = f(x) \quad x \in [a,b], \quad L = \int_a^b \sqrt{1 + f'(x)^2}\,dx$

$C: r = f(\theta) \quad \theta \in [\alpha,\beta], \quad L = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta$

4. 旋轉曲面表面積公式

$C: \begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad t \in [a,b] \Rightarrow \begin{array}{l} A_x = \int_a^b 2\pi|y(t)|\sqrt{x'(t)^2 + y'(t)^2}\,dt \\ A_y = \int_a^b 2\pi|x(t)|\sqrt{x'(t)^2 + y'(t)^2}\,dt \end{array}$

其中 A_x：C 繞 X-軸所得曲面之表面積

$C: y = f(x), \quad x \in [a,b] \Rightarrow A_x = \int_a^b 2\pi|f(x)|\sqrt{1 + f'(x)^2}\,dx$

$A_y = \int_a^b 2\pi|x|\sqrt{1 + f'(x)^2}\,dx$

$C: x = g(y), \quad y \in [c,d] \Rightarrow A_x = \int_c^d 2\pi|y|\sqrt{1 + g'(y)^2}\,dy$

$A_y = \int_c^d 2\pi|g(y)|\sqrt{1 + g'(y)^2}\,dy$

$C: r = f(\theta), \quad \theta \in [\alpha,\beta] \Rightarrow A_x = \int_\alpha^\beta 2\pi|f(\theta)\sin\theta|\sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta$

$$A_y = \int_\alpha^\beta 2\pi \left| f(\theta)\cos\theta \right| \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$$

5. 旋轉體體積

R_I由$\begin{cases} y = g_1(x) \\ y = g_2(x) \\ x = a \\ x = b \end{cases}$所圍，且$g_1(x) \le g_2(x), \forall x \in [a,b]$

$$R_{IX} = \int_a^b \pi g_2(x)^2 dx - \int_a^b \pi g_1^2(x) dx = \int_a^b \pi \left[g_2(x)^2 - g_1(x)^2 \right] dx$$

$$R_{Iy} = \int_a^b 2\pi |x| \left(g_2(x) - g_1(x) \right) dx$$

$R_{Ix} : R_I$ 繞 x-軸旋轉所得體積

$R_{Iy} : R_I$ 繞 y-軸旋轉所得體積

$R_{\Pi x}, R_{\Pi y}$ 比照辦理

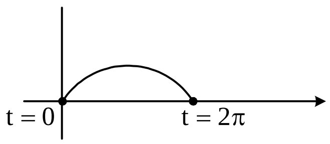
$$R_{\Pi x} = \int_c^d 2\pi |y| (h_2(y) - h_1(y)) dy$$

$$R_{\Pi y} = \int_c^d \pi h_2(y)^2 dy - \int_c^d \pi h_1^2(y) dy = \int_c^d \pi \left[h_2^2(y) - h_1^2(y) \right] dy$$

6. 擺線弧長公式

(1)基本擺線每一拱長 L_1

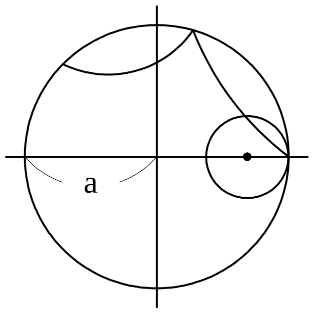
$C: \begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases} t \in [0, 2\pi]$



$L_1 = 8a$

(2)內擺線(n-尖內擺線)

$C: \begin{cases} x = (a-b)\cos\theta + b\cos\dfrac{a-b}{b}\theta \\ y = (a-b)\sin\theta - b\sin\dfrac{a-b}{b}\theta \end{cases}$



$$L_1 = \frac{8(a-b)}{n}$$

$$(\frac{a}{b} = n)$$

(3)外擺線

$$C:\begin{cases} x=(a+b)\cos\theta-b\cos\dfrac{a+b}{b}\theta \\ y=(a+b)\sin\theta-b\sin\dfrac{a+b}{b}\theta \end{cases}$$

$$L_1=\frac{8(a+b)}{n}$$

$$(n=\frac{a}{b})$$

主題一：參數式與極坐標（Ⅱ）

1. $x=\dfrac{t}{1+t}, y=\dfrac{t_2}{1+t}$, 求 $\dfrac{dy}{dx},\dfrac{d^2y}{dx^2}$

$ANS:\dfrac{dy}{dx}=2t+t^2,\dfrac{d^2y}{dx^2}=2(1+t)^3$

註： $x=x(t), y=y(t)\Rightarrow\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{y'(t)}{x'(t)}$

$$\frac{d^2y}{dx^2}=\frac{x'(t)y''(t)-y'(t)x''(t)}{\left[x'(t)^3\right]}$$

2. $x=\int_0^{t-1}\sqrt{x^2+1}\ \ dx, y=\int_0^{t^3}\dfrac{1}{y+3}dy$

$then\quad (i)\dfrac{dy}{dx}\Big|_{x=0}=?\qquad (ii)\dfrac{dy}{dx}\Big|_{t=2}=?$

$ANS:(i)\dfrac{3}{4}(ii)\dfrac{12}{11\sqrt{2}}$

3. $C:\begin{cases} x=a(\cos t+t\sin t) \\ y=a(\sin t-t\cos t)\end{cases} t\in[0,T]$. Find length of C

$ANS{:}=\dfrac{1}{2}aT^2$

4. $C:y=\int_0^x\sqrt{\cos t}\ \ dt\ , x\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$. Find length of C

ANS:4

5. $C:r=a(1+\cos\theta)$ Find length of C

ANS:8a

6. $C:x^{2/3}+y^{2/3}=a^{2/3}$ Find length of C

ANS:6a

7. $C:(\frac{x}{a})^{2/3}+(\frac{y}{b})^{2/3}=1$ (a>0,b>0)

$ANS:\frac{4(a^2+ab+b^2)}{a+b}$

8. $C:\begin{cases}x=a(t-\sin t)\\ y=a(1-\cos t)\end{cases}$ $t\in[0,2\pi]$ Find A_x, A_y

$ANS: A_x=\frac{64}{3}\pi a^2, A_y=16\pi^2a^2$

9. $C: y=\sin x$. Find A_x

$ANS:\pi(\sqrt{2}+\ln(\sqrt{2}+1))$

10. $C: r=a(1+\cos\theta)$. Find A_x

$ANS:\frac{32}{5}\pi a^2$

11. $C: x^{2/3}+y^{2/3}=a^{2/3}$. Find A_x

$ANS:\frac{12}{5}\pi a^2$

12. R:bounded by $x=y^2$, x=4, x-axis. Find the volume of solid generated by rotating R about
(1) x-axis (2) y-axis (3) x=4 (4) y=2

$ANS:(1)8\pi\quad(2)\frac{128}{5}\pi\quad(3)\frac{256}{15}\pi\quad(4)\frac{40}{3}\pi$

主題二：偏微分

(I)是非題

1. $\left(\lim_{(x,y)\to(x_0,y_0)}f(x,y)=l\right)\Rightarrow(\lim_{x\to x_0}(\lim_{y\to y_0}f(x,y))=\lim_{y\to y_0}(\lim_{x\to x_0}f(x,y))=l)$

ANS:No

2. $\begin{pmatrix}\lim_{x\to x_0}(\lim_{y\to y_0}f(x,y))\\ =\lim_{y\to y_0}(\lim_{x\to x_0}f(x,y))\end{pmatrix}\Rightarrow(\lim_{(x,y)\to(x_0,y_0)}f(x,y)\ exist)$

ANS:No

3. $\begin{pmatrix}\lim_{(x,y)\to(x_0,y_0)}f(x,y)\ exist\\ \lim_{x\to x_0}f(x,y)\ exist\\ \lim_{y\to y_0}f(x,y)\ exist\end{pmatrix}\Rightarrow\begin{pmatrix}\lim_{x\to x_0}(\lim_{y\to y_0}f(x,y))\\ \lim_{y\to y_0}(\lim_{x\to x_0}f(x,y))\end{pmatrix}$

ANS:Yes

4. $(f_x(x_0,y_0), f_y(x_0,y_0)$ all exist$)\Rightarrow(f(x,y)\ \ is\ \ continuous\ \ at\ \ (x_0,y_0))$

ANS:No

5. $((f_x(x_0,y_0), f_y(x_0,y_0)$ all exist$)\Rightarrow(f(x,y)\ \ is\ \ differentiable\ \ at\ \ (x_0,y_0))$

ANS:No

6. $\left(f_x(x,y), f_y(x,y)\ \ are\ \ continuous\ \ on\ \ N(x_0,y_0)\right) \Rightarrow \left(f(x,y)\ \ is\ \ differentidable\ \ at\ \ (x_0,y_0)\right)$

   $ANS: Yes$

7. $\left(f_x(x,y), f_y(x,y)\ \ are\ \ continuous\ \ on\ \ N_\delta(x_0,y_0)\right) \Rightarrow \left(f(x,y)\ \ is\ \ continuous\ \ at\ \ (x_0,y_0)\right)$

   $ANS: Yes$

8. $\left(f_{xy}(x_0,y_0), f_{yx}(x_0,y_0)\ \ all\ \ exist\right) \Rightarrow \left(f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0)\right)$

   $ANS: NO$

9. $\left(f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}\right) \Rightarrow \left(f(x,y,z)\ \ is\ \ an\ \ harmonic\ \ function\right)$

   $ANS: NO$

(Ⅱ)計算題

1. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+3y^2}{\sqrt{x^2+3y^2+1}-1} = ?$

   $ANS: 2$

   註:原式= $\lim\limits_{(x,y)\to(0,0)} \dfrac{(x^2+3y^2)(\sqrt{x^2+3y^2+1}+1)}{x^2+3y^2} = ...$

2. $f(x,y) = \begin{cases} \dfrac{\sin(x^2+y^2)}{x^2+y^2} & (x,y)\neq(0,0) \\ 1 & (x,y)=(0,0) \end{cases}$

   Find $f_x(0,0), f_y(0,0)$

   ANS: 0 , 0

   註:利用基本定義式 $f_x(x_0,y_0) = \lim\limits_{x\to x_0} \dfrac{f(x,y_o)-f(x_0,y_o)}{x\to x_0}$

3. $f(x,y) = \begin{cases} \dfrac{xy(x^2+y^2)}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$ Find (i) $f_{xy}(0,0)$ (ii) $f_{yx}(0,0)$

   ANS:(i)-1 (ii)1

4. Let $f(tx,ty,tz) = t^p f(x,y,z)$ then $\dfrac{x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} + z\dfrac{\partial f}{\partial z}}{f(x,y,z)} = ?$

   ANS:p

   註:此爲 Euler theorem for homogeneous function

5. Let $u = u(x,y)$, $v = v(x,y)$ and $u + \ln v = x - y$, $v + \ln u = x + y$, then $\dfrac{\partial u}{\partial x}$ =?

   ANS: $\dfrac{uv-u}{uv-1}$

   註:原式對 x 偏微,可得 $\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial x}$ 之聯立方程式

6. $\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$ are all continuous, and $\frac{\partial f(x,y)}{\partial y}=(\cos x)e^{-y^2}$, f(0,0)=0 then $\lim_{t\to 0}\frac{f(t^5,4t)}{t}=?$

   ANS:4

7. Find the straight line that best fit the data points:(1,1) , (2,3) , (3,6) , where "best" is meant in the sense of the least square.

   ANS: $y=-\frac{5}{3}+\frac{5}{2}x$

8. Let $f(x,y)=xe^y$, then the maximum rate of change of f at the point (2,0) is ____.

   ANS: $\sqrt{5}$

9. The unit normal vector for the surface $z^2=4(x^2+y^2)$ at the point (1,0,2) is ____.

   ANS: $[0,\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}]$

10. The tangent plane for the quadric surface $Ax^2+By^2+Cz^2+D=0$ at the point (x_0,y_0,z_0) is ____.

   ANS: $Axx_0+Byy_0+Czz_0+D=0$

主題三：重積分

（Ⅰ）是非題

1. $f(x,y)=\frac{y}{x}$, $g(x,y)=\frac{x-y}{x+y}$, then f,g are functionally dependent.

   ANS:*NO*

   註: $J(f,g;x,y)=......=0 \quad \forall \quad x+y\neq 0,\ x\neq 0$

2. $f(x,y)=x^2+2xy+y^2+2x+2y$, $g(x,y)=e^xe^y$, then *f, g* are function dependent.

   ANS:*YES*

   註: $J(f,g;x,y)=......=0 \quad \forall \quad x,y\in R^2$

（Ⅱ）計算題

1. $Let\int_0^2 dy\int_0^{2y}(x^2+y)dx=\int_0^4\int_{B(x)}^{C(x)}(x^2+y)dy=A$ the A=? B(x)=? C(x)=?

   $ANS: A=16,\quad B(x)=\frac{x}{2},\ \ C(x)=2$

2. Find the volume bounded by the cylinders: $x^2+z^2=a^2$, $x^2+y^2=a^2$

   ANS: $\frac{16}{3}a^3$

3. Find the surface area of the region bounded by the cylinders:
   $x^2+z^2=a^2$, $x^2+y^2=a^2$
   ANS: $16a^2$

4. The area of the region enclosed by the curve : $r=2\sin 3\theta$
   ANS: π

5. $\int_0^a \int_0^x \sqrt{x^2+y^2}\,dydx = ?$

$ANS: \frac{1}{6}a^3(\sqrt{2}+\ln(\sqrt{2}+1))$

6. $\int_0^\infty e^{-x^2}dx = ?$

$ANS: \frac{\sqrt{\pi}}{2}$

7. $\int_{-\infty}^\infty e^{-x^2}dx = ?$

$ANS: \sqrt{\pi}$

8. $\int_{-\infty}^\infty e^{-\frac{x^2}{2}}dx = ?$

$ANS: \sqrt{2\pi}$

9. $\int_0^\infty x^2 e^{-x^2}dx = ?$

$ANS: \frac{\sqrt{\pi}}{4}$

10. $\int_1^\infty e^{-\frac{(x-1)^2}{6}}dx = ?$

$ANS: \frac{\sqrt{6\pi}}{2}$

11.Find $\iint_\Omega e^{\frac{y+x}{y-x}}dxdy$, where Ω is the region bounded by x=0, y=0, y=1+x, y=3+x

*ANS:*1

註：令 u= y - x , υ = y + x, 採用 Jacobian 轉換。

主題四：微分方程式

1. Let f(x) = 1 + $\int_0^x t^3\ f(t)$dt, then f(1) = ?

$ANS: \sqrt[4]{e}$

註：＜STEP＞將積分方程改成微分方程

原式對 x 微分得： $f'(x)=x^3f(x)$ --------(1)

原式令 x=0 $f(0)=1$ -----------------(2)

(1) 得： $\ln|f(x)|=\frac{1}{4}x^4+c$

$\Rightarrow |f(x)| = e^{\frac{1}{4}x^4+c} = ke^{\frac{1}{4}x^4}$ (k=e^0>0)

$\therefore\ f(x)=ke^{\frac{1}{4}x^4}$ 或 $f(x)=-ke^{\frac{1}{4}x^4}$

由 $f(0)=1$ 得知宜取 $f(x)=ke^{\frac{1}{4}x^4}$，且 k=1

$\therefore$ 得 $f(x)=e^{\frac{1}{4}x^4}$

$\therefore$ $f(1)=e^{\frac{1}{4}}=\sqrt[4]{e}$

2. Let $\frac{dy}{dx}=y^2e^{2x}, y(0)=2,$ then y(x) = ?

ANS：$\frac{2}{2-e^{2x}}$

3. Let $\left(e^{-y}-2x\right)dx-\left(xe^{-y}+\sin y\right)dy=0$ Find the solution

ANS：$xe^{-y}-x^2+\cos y+c=0$

4. Let F be a function of x,y, and $\frac{\partial F}{\partial X}=e^{-y}-2x, \frac{\partial F}{\partial Y}=-\left(xe^{-y}+\sin y\right)$ and F(0,0)=2, Then F(x,y)= ?

ANS：F(x,y)= $xe^{-y}-x^2+\cos y+1$

5. Let (4xy + 2x)dx + g(x,y)dy = 0 be a exact differential equation. Then g(x,y) =____.

ANS：$2x^2+k(y)$

註：根據 exact O,D,E 之義得

$$\frac{\partial}{\partial y}(4xy+2x)=\frac{\partial}{\partial x}g(x,y)$$

$$\Rightarrow 4x=\frac{\partial}{\partial x}g(x,y)$$

$$\therefore g(x,y)=2x^2+k(y)$$

6. Let $y=a\sin x+b\cos x$ be a solution for the differential equation $y''-y'-by=\sin x$,then a = ? b = ?

ANS：$a=-\frac{7}{50}, b=\frac{1}{50}$

7. Find the general solution for $y''-y'-by=\sin x$

ANS：$y=c_1e^{3x}+c_2e^{-2x}+\left(-\frac{7}{50}\right)\sin x+\frac{1}{50}\cos x$

8. Let $f(x)=2+\int_0^x \frac{f(t)}{(t+2)(t+3)}dt$ for $x\geq 0$, then $f(1)=?$

ANS：$\frac{9}{4}$

註：比照第 1 題

主題五：線積分

1. Let curve C： $x=2\cos t, y=2\sin t \quad t\in[0,2\pi]$ Then the line integral $\int_c \left(x^2y+2\right)dS=?$

ANS： 8π

註： $\int_c \left(x^2y+2\right)dS=\int_0^{2\pi}\left(x^2y+2\right)\sqrt{x^1(t)^2+y^1(t)^2}\,dt$

$=\int_0^{2\pi}\left[(2\cos t)^2\cdot(2\sin t)+2\right]\sqrt{(-2\sin t)^2+(2\cos t)^2}\,dt$

$=\int_0^{2\pi}\left(8\cos^2 t\sin t+2\right)\cdot 2dt$

$=\left[16\left(-\frac{1}{3}\cos 3t\right)+4t\right]_0^{2\pi}=8\pi$

2. Find the line integral $\int_c \left(x^2y+2\right)dS$, Where C is the path $x=2\cos t, y=2\sin t$ in a clockwise direction from (0,2) to (2,0)

ANS： $-\frac{16}{3}-2\pi$

註：由題意知

$$\int_c \left(x^2y+2\right)dS=\int_{\frac{\pi}{2}}^{0}\left[(2\cos t)^2(2\sin t)+2\right]\sqrt{(-2\sin t)^2+(2\cos t)^2}\,dt=-\frac{16}{3}-2\pi$$

3. Find the line integral $\int_c \left(x^2y+2\right)dS$ where C is the curve $x^2+y^2=4$ in a clockwise direction from (0,2) to (2,0)

ANS： $-\frac{16}{3}-2\pi$

4. Let $\vec{F}(x,y,z)=xy\,\hat{i}+x^2z\,\hat{j}+xyz\,\hat{k}$ C： $\vec{r}(+)=e^t\,\hat{i}+e^{-t}\,\hat{j}+t^2\,\hat{k}$, $0\le t\le 1$ Find the line integral $\int_c \vec{F}\cdot d\vec{r}$

ANS：2

註： $\int_c \vec{F}\cdot d\vec{r}=\int_c xydx+x^2zdy+xyzdz$

$=\int_0^1\left(e^t\cdot e^{-t}\right)e^tdt+\left(e^t\right)^2\left(t^2\right)(-dt)+\left(e^t\cdot e^{-t}\cdot t^2\right)2tdt=2$

5. Let $\vec{F}(x,y,z)=xy\,\hat{i}+x^2z\,\hat{j}+xyz\,\hat{k}$ Find the line integral $\int_c \vec{F}\cdot d\vec{r}$ along the path C：

$$C:\begin{cases} x=e^t \\ y=e^{-t}, 0\le t\le 1 \\ z=t^2 \end{cases}$$

ANS：2

註：解法與第 4 題同，只不過 C 的表示方法不同

6. Let $\vec{F}(x,y,z)=xy\,\hat{i}+x^2z\,\hat{j}+xyz\,\hat{k}$ Find the line integral $\int \vec{F}\cdot d\vec{r}$ along the path C：$[0,1]\to R^3$ $C(t)=\left[e^t,e^{-t},t^2\right]$

7. (1)Is the line integral $\int_c e^{yz}dx+xze^{yz}dy+xye^{yz}dz$ path independent ?

(2)If yes , find $\int_{(0,0,0)}^{(2,3,4)} e^{yz}dx+xze^{yz}dy+xye^{yz}dz$

註：(1)令 $P(x,y,z)=e^{yz}$， $Q(x,y,z)xye^{yz}$， $R(x,y,z)=xye^{yz}$，

可得 $\dfrac{\partial P}{\partial y}=\dfrac{\partial Q}{\partial x}$ ， $\dfrac{\partial Q}{\partial z}=\dfrac{\partial R}{\partial y}$ ， $\dfrac{\partial P}{\partial z}=\dfrac{\partial R}{\partial x}$. ∴得知 Path indep.

(2)令 $\nabla\phi=[P,Q,R]\Rightarrow \dfrac{\partial\phi}{\partial x}=P,\dfrac{\partial\phi}{\partial y}=Q,\dfrac{\partial\phi}{\partial z}=R \Rightarrow \phi=xe^{yz}+C$.

$\therefore \int_{(0,0,0)}^{(2,3,4)}\cdots = xe^{yz}\Big|_{(0,0,0)}^{(2,3,4)}=2e^{12}$

8. $\int_{(0,0,0)}^{(2,3,4)}(e^{yz}dx+xze^{yz}dy+xye^{yz}dz)=?$

ANS： $2e^{12}$

註：易知本題屬於 Path indep. line integral. 比照第 7 題解之

9. Given $\vec{F}(x,y,z)=\left(z^3+3x^2y\right)\hat{I}+\left(x^3+3y^2z\right)\hat{J}+\left(y^3+3z^2x\right)\hat{K}$

(1)Prove that $\vec{F}$ is conservative (i.e. $\int_c \vec{F}\cdot d\vec{r}$ is path indep.)

(2)Find the potential function ϕ of $\vec{F}$ (i.e. $\nabla\phi=\vec{F}$)

(3)Find $\int_{(0,0,0)}^{(1,1,1)}\vec{F}\cdot d\vec{r}$

ANS：(2) $\phi=xz^3+x^3y+y^3z+k$ (3) 3

10. Use the Green Thm to find the line integral $\int_c x^2ydx+3xydy$ where C is the positive oriented simple closed curve determined by the graphs of $y=x^2$, $y=\sqrt{x}$

ANS： $\dfrac{51}{140}$

註： $\int x^2ydx+3xydy=\iint\limits_R\left[\dfrac{\partial}{\partial x}(3xy)-\dfrac{\partial}{\partial y}(x^2y)\right]dA$

$=\iint\limits_R(3y-x^2)dA=\int_0^1\int_{x^2}^{\sqrt{x}}(3y-x^2)dydx=\dfrac{51}{140}$