高雄醫學大學八十九學年度學士後醫學系招生考試試題

科目：微積分　　　　　　　　**考試時間：八十分鐘**

1. Label each statement as true or false. (20%)

(a) If $\lim_{x\to a} f(x) = A$, $\lim_{x\to a} g(x) = B$, and $f(x) < g(x)$ for all $x \in R$, then $A < B$.

(b) If $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist, then $\lim_{x\to a} f(x)$ exists.

(c) If $f(1) > 0$ and $f(3) < 0$, there exists a number c between 1 and 3 such that $f(c)=0$.

(d) $\frac{d}{dx}(\tan^2 x) = \frac{d}{dx}(\sec^2 x)$

(e) The average value of the function $f(x)=1+x^2$ on the interval [-1, 2] is 2.

(f) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges.

(g) If $a_n > 0$ and $\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) < 1$, then $\lim_{n\to\infty} a_n = 0$

(h) Let $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$, then f is differentiable at $x = 0$

(i) If f has continuous first partial derivatives in a neighborhood of the point (a, b), then f can have all of its directional derivatives at (a, b).

(j) If fx and fy both exist and equal at (a, b), then f is continuous at (a, b).

2. Evaluate each of the following limits, if it exists. (20%)

(a) $\lim_{x\to 0} \frac{|2x-1|-|2x+1|}{x}$　(b) If [x] denotes the greatest integer function, find $\lim_{x\to\infty} \frac{x}{[x]}$.

(c) $\lim_{x\to\infty} \sqrt[n]{(89)^n + (90)^n + (91)^n + \cdots\cdots + (2000)^n}$.　(d) $\lim_{x\to 3} \left(\frac{x}{x-3}\int_3^x \frac{\sin t}{t} dt\right)$.

3. Evaluate the following integrals. (20%)

(a) $\int_0^1 \sqrt{x}\ln x^2 dx$

(b) $\int_{-2}^{-1} \frac{2x^2+4x+2}{x^3+2x^2+2x} dx$

(c) $\int_0^1 \int_y^1 e^{-x^2} dxdy$

(d) $\iint_D \sqrt{1-x^2-y^2}\, dA$, where D is the disk $x^2+y^2 \le 1$

4. If f is a quadratic function such that f(0) =1 and $\int \frac{f(x)}{x^2(x+1)^3} dx$ is a rational function, find the value of f' (0). (10%)

5. Find all functions f such that f' is continuous and $(f(x))^2 = 100 + \int_0^x (f(t))^2 + (f'(t))^2 dt$ for all real x. (10%)

6. Find the maximum and minimum values of x – y + z if (x, y, z) lies on the sphere $x^2 + y^2 + z^2 = 1$. (10%)

7. Find the area of the bounded region enclosed by curves $y = x^3 - 1$ and $y = 3x^2 - 2x - 1$. (10%)

89 年學士後西醫試題解答

1.(a) false

理由：例如 $f(x)=\dfrac{x^2+1}{x^2}$ $g(x)=\dfrac{x^2+2}{x^2}$

$f(x)<g(x)\ \forall x\in R$

$\lim\limits_{x\to\infty} f(x)=1,\ \lim\limits_{x\to\infty} g(x)=1$

(b) false

理由： $\lim\limits_{x\to\infty} f(x)=1,\ \lim\limits_{x\to a^+} f(x)=2$

$\Rightarrow \lim\limits_{x\to a} f(x)$ **不存在**

(c) false

理由：必須 f(x) ε cont. on[1, 3]**才能保証存在一個** C(**介乎** 1 **與** 3 **之間)，使得** f(c) = 0

例如： $f(x)=\dfrac{1}{x-2}$ **即是**

(d) true

$$\frac{d}{dx}(\tan^2 x)=2\tan x\sec^2 x$$

$$\begin{aligned}\frac{d}{dx}(\sec^2 x)&=2\sec x\cdot\sec x\tan x\\&=2\sec^2 x\tan x\end{aligned}$$

(e) true

$$\begin{aligned}<f>&=\frac{\int_{-1}^{2} f(x)dx}{2-(-1)}=\frac{\int_{-1}^{2}(1+x^2)dx}{3}\\&=\frac{6}{3}=2\end{aligned}$$

(f) false

例如： $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ conv　　(**交錯級數特性**)

$\sum_{n=1}^{\infty} a_n^{\ 2} = \sum_{n=1}^{\infty} \frac{1}{n}$ div　　(P－**級數特性或調和級數特性**)

(g) true

理由： $\sum_{n=1}^{\infty} a_n$ **：正項級數且** $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ **為收斂** (**根據比值審斂法**)

$\therefore \lim_{n\to\infty} a_n = 0$ (**收斂級數之必要條件**)

(h) true

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^2 \sin\frac{1}{x} - 0}{x - 0}$$

$$= \lim_{x\to 0} x \sin\frac{1}{x} = 0 \quad \therefore f'(0)\text{存在}$$

$\therefore$ f(x)**在** $x = 0$ **處可微**

(i) true

理由：一階偏導數在(a, b)**處連續，下述定理成立：**

$$D_{\hat{e}} f(a,b) = [fx(a,b), fy(a,b)] \cdot \hat{e}$$

(j) false

例如 $f(x,y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

顯然 fx (0,0) = 0　fy (0,0) = 0

(fx(0,0) = fy (0,0))

但 f(x,y)**在**(0,0)**處不連續**

2. (a)

$$1^\circ \lim_{x\to 0}\frac{|2x-1|-|2x+1|}{x}=\lim_{x\to 0^+}\frac{-(2x-1)-(2x+1)}{x}$$

$$=\lim_{x\to 0}\frac{-4x}{x}=-4$$

$$2^\circ \lim_{x\to 0^-}\frac{|2x-1|-|2x+1|}{x}=\lim_{x\to 0^-}\frac{-(2x-1)-(2x+1)}{x}$$

$$=\lim_{x\to 0^+}\frac{-4x}{x}=-4$$

$$\therefore \lim_{x\to 0}\frac{|2x-1|-|2x+1|}{x}=-4$$

(b)

$$x-1\le[x]\le x \quad \forall x\in R$$

$$\Rightarrow \frac{1}{x}\le\frac{1}{[x]}\le\frac{1}{x-1} \quad \forall x>100$$

$$\Rightarrow \frac{x}{x}\le\frac{x}{[x]}\le\frac{x}{x-1} \quad \forall x>100$$

$$\lim_{x\to\infty}\frac{x}{x}=1,\ \lim_{x\to\infty}\frac{x}{x-1}=1$$

$$\therefore \lim_{x\to\infty}\frac{x}{[x]}=1$$ (**夾擊原理**)

(c)

$$(2000)^n\le(89)^n+(90)^n+\cdots\cdots+(2000)^n\le(2000)^n+\cdots\cdots+(2000)^n \ \forall n\in N$$

$$\Rightarrow (2000)^n\le(89)^n+(90)^n+\cdots\cdots(2000)^n\le(2000-88)\cdot(2000)^n$$

$$\Rightarrow \sqrt[n]{(2000)^n}\le\sqrt[n]{(89)^n+(90)^n+\cdots\cdots(2000)^n}\le\sqrt[n]{(1912)+(2000)^n}$$

$$\lim_{n\to\infty}\sqrt[n]{(2000)^n}=2000,\ \lim_{n\to\infty}\sqrt[n]{(1912)\cdot(2000)^n}$$

$$=\lim_{n\to\infty}\sqrt[n]{1912}\cdot 2000=2000$$

$$(\because \lim_{n\to\infty}\sqrt[n]{1912}=1)$$

$$\therefore \lim_{n\to\infty}\sqrt[n]{(89)^n+(90)^n+\cdots\cdots(2000)^n}=2000$$

(d)

$$\lim_{x\to 3}\frac{3}{x-3}\int_3^x \frac{\sin t}{t}dt\ (\infty\cdot 0\text{型})$$

$$=\lim_{x\to 3}\frac{x\int_3^x \frac{\sin t}{t}dt}{x-3}(\frac{0}{0}\text{型})$$

L'HR

$$=\lim_{x\to 3}\frac{1\cdot\int_3^x \frac{\sin t}{t}dt+x\cdot\frac{\sin x}{x}}{1}$$

$$=\lim_{x\to 3}[\int_3^x \frac{\sin t}{t}dt+\sin x]=\sin 3$$

3. (a) $f(x)=\sqrt{x}\ln x^2 \varepsilon \text{cont.on } (0,1)$

$$\therefore \int_0^1 \sqrt{x}\ln x^2 dx=\lim_{t\to 0^+}\int_t^1 \sqrt{x}\ln x^2 dx$$

$$=\lim_{t\to 0^+}[\frac{2}{3}x^{3/2}\ln x^2-\frac{8}{9}x^{3/2}]\Big|_t^1$$

$$=\lim_{t\to 0^+}\left\{[0-\frac{8}{9}]-[\frac{2}{3}t^{3/2}\ln t^2-\frac{8}{9}t^{3/2}]\right\}$$

$$=-\frac{8}{9}+\lim_{t\to 0^+}[\frac{8}{9}t^{3/2}-\frac{4}{3}t^{3/2}\ln t]$$

$$=-\frac{8}{9}+\lim_{t\to 0^+}\frac{4}{3}t^{3/2}[2-\ln t]\ (0\cdot\infty\text{型})$$

$$=-\frac{8}{9}+\frac{4}{3}\lim_{t\to 0^+}\frac{2-\ln t}{t^{-3/2}}(\frac{\infty}{\infty}\text{型})$$

L'HR

$$=-\frac{8}{9}+\frac{4}{3}\lim_{t\to 0^+}\frac{-\frac{1}{t}}{(-\frac{3}{2})t^{-5/2}}$$

$$=-\frac{8}{9}+\frac{4}{3}\lim_{t\to 0^+}(-\frac{2}{3})(-1)t^{3/2}=-\frac{8}{9}$$

(b)

$$I=\int_{-2}^{-1}\frac{2x^2+4x+2}{x^3+2x^2+2x}=\int_{-2}^{-1}\frac{2x^2+4x+2}{x(x^2+2x+2)}dx$$

$$=\int_{-2}^{-1}[\frac{1}{x}+\frac{x+2}{x^2+2x+2}]\,dx$$

$$=[\ln|x|+\frac{1}{2}\ln|\ x^2+2x+2|+\tan^{-1}(x+1)]\,|_{-2}^{-1}$$

$$=\ln|-1|+\frac{1}{2}\ln|\ (-1)^2+2(-1)+2|+\tan^{-1}$$

$$(-1+1)-[\ln|-2|+\frac{1}{2}\ln|(-2)^2+2(-2)+2|+\tan^{-1}(-2+1)]$$

$$=[0+\frac{1}{2}\cdot 0+0]-[\ln 2+\frac{1}{2}\ln 2+\tan^{-1}(-1)]$$

$$=-(\frac{3}{2}\ln 2-\frac{\pi}{4})=\frac{\pi}{4}-\frac{3}{2}\ln 2$$

(c)

$$I=\int_0^1\int_y^1 e^{-x^2}dxdy$$

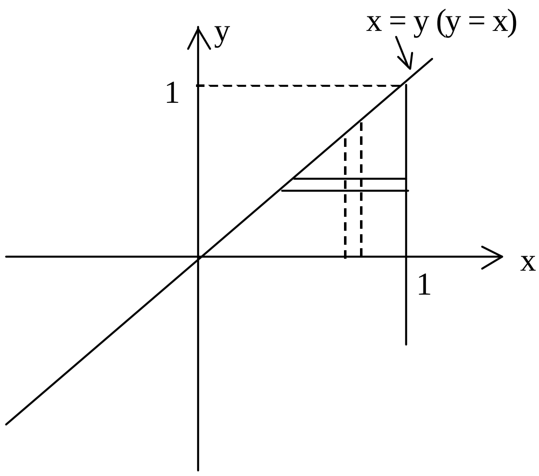
$$=\int_0^1[\int_0^x e^{-x^2}dy]\,dx$$

$$=\int_0^1[ye^{-x^2}\big|_0^x]\,dx$$

$$=\int_0^1[xe^{-x^2}-0]\,dx$$

$$=\int_0^1 xe^{-x^2}dx=(-\frac{1}{2}e^{-x^2})\Big|_0^1$$

$$=\frac{1}{2}(1-e^{-1})$$



(d)

$$I=\iint_D \sqrt{1-x^2-y^2}\ dA \quad D: x^2+y^2\le 1$$

令 $x=r\cos\theta$, $y=r\sin\theta$

$r\in[0,\infty]$, $\theta\in[0,2\pi]$

$$\Rightarrow I=\int_0^{2\pi}[\int_0^1\sqrt{1-r^2}\cdot rdr]\,d\theta$$

$$=[\int_0^{2\pi}d\theta]\cdot[\int_0^1\sqrt{1-r^2}rdr]$$

$$=2\pi\cdot(-\frac{1}{3})(1-r^2)^{3/2}\Big|_0^1$$

$$=2\pi\cdot(-\frac{1}{3})[0-1]=\frac{2}{3}\pi$$

4.**根據題意：**f(x)**為二次多項式，且** f(0) = 1，**可令** $f(x) = ax_2 + bx + 1$

$$I=\int\frac{f(x)}{x^2(x+1)^3}dx=\int\frac{ax^2+bx+1}{x^2(x+1)^3}dx$$

$$=\int[\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}+\frac{D}{(x+1)^2}+\frac{E}{(x+1)^3}]\,dx$$

$$=\int[\frac{B}{x^2}+\frac{D}{(x+1)^2}+\frac{E}{(x+1)^3}]\,dx$$

($\because$ I **為有理式**)

故 $\int\frac{A}{x}dx$ **及** $\int\frac{C}{x+1}dx$ **均為** 0

$$\Rightarrow B(x+1)^3+Dx^2(x+1)+Ex^2=ax^2+bx+1$$

$$\Rightarrow (B+D)x^3+(3B+D+E)x^2+3Bx+B$$

$$=ax^2+bx+1$$

$$\Rightarrow B=1\Rightarrow b=3b=3$$

$$\therefore f(x)=ax^2+3x+1$$

$$f'(s)=2ax+3$$

$$\therefore f'(0)=3$$

5. $f(x)^2 = 100 + \int_0^x [(f(t))^2 + (f'(t))^2]\, dt \cdots\cdots (1)$

(1)式令 $x = 0 \Rightarrow f(0)^2 = 100 \Rightarrow f(0) = 10$ 或-10

(1)式對 x 微分

$$2f(x)\cdot f'(x) = f(x)^2 + f'(x)^2$$
$$\Rightarrow f(x)^2 - 2f(x)\cdot f'(x) + f'(x)^2 = 0$$
$$\Rightarrow (f(x) - f'(x))^2 = 0 \Rightarrow f(x) - f'(x) = 0$$
$$\Rightarrow \frac{df}{dx} = f \cdots\cdots (2)$$

(2) $\frac{df}{f} = dx \Rightarrow \ln|f| = x + c$

$$\Rightarrow |f| = e^{x+c} \Rightarrow f(x) = \pm e^{x+c} = \pm e^c \cdot e^x$$

情況(i)： $f(0) = -10 \Rightarrow f(x) = -e^c e^x$

(取“+”號) $\Rightarrow 10 = e^c \cdot e^0 \Rightarrow e^c = 10$

$\therefore f(x) = 10e^x$

情況(ii)： $f(0) = -10 \Rightarrow f(x) = -e^c e^x$

(取“-”號) $\Rightarrow -10 = -e^c e^0 \Rightarrow e^c = 10$

$\therefore f(x) = -10e^x$

6.令 $f(x, y, z) = x - y + z$

$g(x, y, z) = x^2 + y^2 + z^2 - 1$

$W(x, y, z, \lambda) = f(x, y, z) + \lambda g(x, y, z)$

$= x - y + z + \lambda(x^2 + y^2 + z^2 - 1)$

根據 Lagromge mulfiplier method

f (x, y, z)呈極值之必要條件：

$Wx = Wy = Wz = W\lambda = 0$

$$\Rightarrow \begin{cases} 1 + 2\lambda x = 0 \cdots\cdots\cdots\cdots\cdots\cdots 1 \\ -1 + 2\lambda y = 0 \quad \cdots\cdots\cdots\cdots 2 \\ 1 + 2\lambda z = 0 \quad \cdots\cdots\cdots\cdots\cdots 3 \\ x^2 + y^2 + z^2 - 1 = 0 \cdots\cdots 4 \end{cases}$$

1 得 $x = \frac{-1}{2\lambda}$ 　 $2\, y = \frac{1}{2\lambda}$ 　 $3\, z = \frac{-1}{2\lambda}$

將上述 x, y, z **代入** 4

$$(-\frac{1}{2\lambda})^2+(\frac{1}{2\lambda})^2+(\frac{-1}{2\lambda})^2=1\Rightarrow\frac{3}{4\lambda^2}=1$$

$$\Rightarrow\lambda^2=\frac{3}{4}\Rightarrow\lambda=\frac{\sqrt{3}}{2}\text{或}-\frac{\sqrt{3}}{2}$$

(i) $\lambda=\frac{\sqrt{3}}{2}\Rightarrow(x,y,z)=(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}})$

$$\Rightarrow f(x,y,z)=-\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}}=-\frac{1}{\sqrt{3}}$$

(ii) $\lambda=-\frac{\sqrt{3}}{2}\Rightarrow(x,y,z)=(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})$

$$\Rightarrow f(x,y,z)=\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}=\sqrt{3}$$

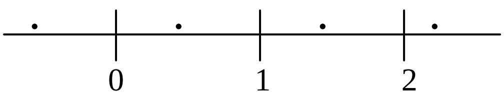
$\therefore$ **得最大值為** $f(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})=\sqrt{3}$

最小值為 $f(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}})=-\sqrt{3}$

7.**曲線** $y=x^3-1$ **與** $y=3x^2-2x-1$ **之交點出現在** $x=0, 1, 2$ **處**

(**令** $x^3-1=3x^2-2x-1\Rightarrow x=1, 1, 2$)

$\therefore$**得兩曲線所圍面積** A**：**



$$A=\int_0^2\left|(x^3-1)-(3x^2-2x-1)\right|dx$$

$$=\int_0^2\left|x(x-1)(x-2)\right|dx$$

$$=\int_0^1\left|x(x-1)(x-2)\right|dx+\int_1^2\left|x(x-1)(x-2)\right|dx$$

$$=\int_0^1 x(x+1)(x-2)\,dx+\int_1^2[-x(x-1)(x-2)]\,dx$$

$$=\int_0^1(x^3-3x^2+2x)\,dx+\int_1^2[-(x^3-3x^2+2x)]\,dx$$

$$=(\frac{1}{4}x^4-x^3+x^2)\Big|_0^1-(\frac{1}{4}x^4-x^3+x^2)\Big|_1^2$$

$$=(\frac{1}{4}-0)-(-\frac{1}{4})=\frac{1}{2}$$