高雄醫學大學九十學年度學士後醫學系招生考試試題

科目：微積分 **考試時間：八十分鐘**

一、是非題 （共有 5 題，每一題 2 分，共 10 分。請將答案依 a, b, c…次序寫出 T 或 F）

a. If a function *f* is increasing in $(1-\in, 1)(\in>0)$ and decreasing in $[1, 1+\in]$, then *f* has a local maximum at x = 1.

b. Let *f* be a nonnegative continuous function on $\Re$. If $\int_0^\infty f(x)\,dx<\infty$, than *f* is bounded above.

c. There exists a constant α such that the function *f* defined by $f(x)=\begin{cases} x^2, & \text{if } x\le 1 \\ \alpha\sqrt{x}, & \text{if } x>1 \end{cases}$ is differentiable everywhere.

d. The series $\sum_{n=1}^{\infty} a_n x^n$ and the series $\sum_{n=1}^{\infty} na_n x^{n-1}$ have the same radius of convergence.

e. Let g be a positive continuous function defined on $\Re$ satisfying $\int_0^\infty g(x)dx=\infty$ and let $\Omega=\{(x,y): 0\le x<\infty, 0\le y\le g(x)\}$. Then the volume of solid generated by revolving Ω about the x-axis must be also infinity.

二、填充題 （共有 5 個空格，每一空格 6 分，共 30 分。請將答案依 a, b, c…次序寫出，不需演算過程）

a. $\lim_{x\to 0}\frac{x\cos x-\sin x}{x-\sin x}=$____________.

b. $\lim_{n\to\infty}\sum_{k=1}^{n}\ln\sqrt[n]{1+\frac{k}{n}}=$____________.

c. Let $H(x)=\int_0^{x^2}\frac{dt}{1+t^3}$ and $L(x)=\int_0^{x}\frac{dt}{1+t^3}$, then $H'(2)-L'(4)=$____________.

d. If $x=t-t^2$, $y=t+t^2$, then $\frac{d^2y}{dx^2}\Big|_{t=1}=$____________.

e. The tangent plane to $x^2+2y^2+z^2=7$ at (1,1,2) is ____________.

三、Let f(x) be a continuous and decreasing function, and let g(x) be the inverse function of f.

If f(2) = 1, f(4) = 0 and $\int_2^4 f(x)dx = 1$, find $\int_0^1 g(x)dx$. （10%）

四、Find all functions f that satisfy the integral equation $f(x) = 2001 + \int_0^x f(t)\cos t\, dt$. （10%）

五、A rectangular wooden box with an open top is to contain $500cm^3$. Ignoring the thickness of the wood, how is the box to be constructed so as to use the smallest amount of wood？ （10%）

六、Find the tangent line to the curve $1 + x^y = y^x$ ($x>0$, $y>0$) at the point P (1,2). （10%）

七、Evaluate the following integrals

(a) $\int_0^\pi \int_x^\pi \frac{\sin y}{y} dydx$. （5%）

(b) $\frac{1}{\pi}\iint_{R^2} \frac{1}{(1+x^2+y^2)^{2001}} dxdy$. （5%）

八、(a) Let A and B be two points in $\Re^2$ and Γ be a curve with A and B as its end points.

Show that the line integral $\int_\Gamma (2x^2+4xy)dx + (2x^2-y^2)dy$ is independent of the choice of the path Γ with end points A and B. （5%）

(b) Evaluate the line integral $\int_{(0,0)}^{(1,1)} (2x^2+4xy)dx + (2x^2-y^2)dy$. （5%）

90 年學士後西醫試題解答

徐　中老師解題

一、是非題

a. T　　b. T

c. F

理由： $f(x)\in \text{diff at } x=1 \Rightarrow f'(1)\text{exist}$

$$\Rightarrow f_-'(1)=f_+'(1)\Rightarrow \lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}$$

$$\text{i.e.} \lim_{x\to 1}\frac{x^2-1^2}{x-1}=\lim_{x\to 1^+}\frac{\alpha\sqrt{x}-1}{x-1}$$

$$\Rightarrow 2=\lim_{x\to 1^+}\frac{\alpha\sqrt{x}-1}{x-1}\Rightarrow \lim_{x\to 1^+}(\alpha\sqrt{x}-1)=0$$

$$\Rightarrow \alpha=1\Rightarrow \lim_{x\to 1^+}\frac{\alpha\sqrt{x}-1}{x-1}=\lim_{x\to 1^+}\frac{\sqrt{x}-1}{x-1}=\frac{1}{2}\ (\text{矛盾})$$

d. T

e. F

理由：例如 $g(x)=\dfrac{1}{x+1}$

則 $\int_0^\infty g(x)dx\to+\infty$

而 $V_x=\int_0^\infty \pi g(x)^2dx=\int_0^\infty \pi\left(\dfrac{1}{x+1}\right)^2dx\in \text{conV}$

二、填充題

a. - 2　　b. 2ln 2-1　　c. $\dfrac{3}{65}$　　d. - 4　　e. x + 2y + 2z – 7 = 0

三、

$I=\int_0^1 g(x)dx$

令 $y=g(x)$ 則 $x=f(y)$, $dx=f'(y)dy$

則 $I=\int_4^2 y\cdot f'(y)dy$

$=\int_4^1 xf'(x)dx$ (dummy variables 原理)

令 $U=x \quad dv=f'(x)dx$

$du=1\cdot dx \quad v=f(x)$

$\Rightarrow I=xf(x)\Big|_4^2 \int_4^2 f(x)dx$

$=2f(2)-4f(4)+\int_2^4 f(x)dx$

$=2\cdot 1-4\cdot 0+1=3$

四、 $f(x)=2001+\int_0^x f(t)\cos t dt \cdots\cdots(1)$

$\frac{d}{dx}: f'(x)=f(x)\cos x$

i.e. $\frac{df}{dx}=f\cos x \quad (f: f(x))$

$\Rightarrow \frac{1}{f}df=\cos x dx$

$\Rightarrow \ln|f(x)|=\sin x+c$

$\Rightarrow |f(x)|=e^{\sin x+c}=e^c\cdot e^{\sin x}$

$=ke^{\sin x}$ (其中 $k=e^c>0$)

$\Rightarrow f(x)=ke^{\sin x}$ 或 $f(x)=-ke^{\sin x}\cdots\cdots(2)$

(1)中，令 $x=0\Rightarrow f(0)=2001\cdots\cdots(3)$

(2)(3)可易知 $f(x)=ke^{\sin x}\cdots\cdots(3)$

且由(2)(3)得 $k=2001$

∴得 $f(x)=2001e^{\sin x}$

五、依題可繪圖如示,

其中盒頂為空,

於是所需材料面

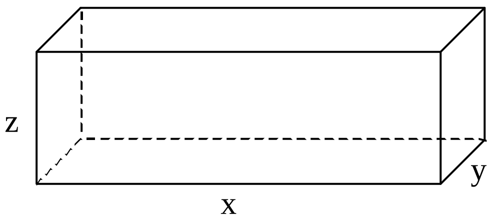
積可設 $f(x,y,z) = xy + 2xz + 2yz$

其中 $xyz = 500$

並令 $g(x,y,z) = xyz - 500$

$\omega = f(x,y,z) + \lambda g(x,y,z)$

$= xy + 2xz + 2yz + \lambda(xyz - 500)$



根據 Lagrornge' s multiplier method.

$F(x,y,z)$ **呈極值之必要條件：**

$\omega_x = 0$, $\omega_y = 0$, $\omega_z = 0$, $\omega_\lambda = 0$

$$\Rightarrow \begin{cases} y + 2z + \lambda yz = 0 \cdots\cdots(1) \\ x + 2z + \lambda xz = 0 \cdots\cdots(2) \\ 2x + 2y + \lambda xy = 0 \cdots\cdots(3) \\ xyz - 500 = 0 \cdots\cdots(4) \end{cases}$$

(1)(2)(3)**可得**

$$\frac{1}{z} + \frac{2}{y} = -\lambda \cdots\cdots(5)$$

$$\frac{1}{z} + \frac{2}{x} = -\lambda \cdots\cdots(6)$$

$$\frac{2}{y} + \frac{2}{x} = -\lambda \cdots\cdots(7)$$

(5)(6)(7)**可得** $x = -\frac{4}{\lambda}, y = -\frac{4}{\lambda}, z = -\frac{2}{\lambda}$，**將** x, y, z **代入**(4)**可得** $\lambda = \frac{-2}{5}$，**於是**

$x = 10$, $y = 10$, $z = 5$ **所以所需最少材面為** $f(10, 10, 5) = 300$

i.e. **長寬約** 10 cm, **而高為** 5 cm

六、

易知 P (1, 2) **在曲線** $1 + y^x = x^y$ **上**

首先求切線斜率 $m = \left.\frac{dy}{dx}\right|_{\substack{x=1\\y=2}}$ **：**

$$1 + x^y = y^x$$

$$\frac{d}{dx}: x^y\,[(\ln x)\cdot\frac{dy}{dx}+\frac{y}{x}] = y^x\,[\ln y+\frac{x}{y}\frac{dy}{dx}]$$

令 (x, y) = (1, 2) **代入上式中**

$$1^2\cdot[\ln(1)\cdot\frac{dy}{dx}+\frac{2}{1}] = 2^1\,[\ln 2+\frac{1}{2}\frac{dy}{dx}]$$

$$\Rightarrow 2 = 2\ln 2+\frac{dy}{dx}$$

$$\therefore \left.\frac{dy}{dx}\right|_{\substack{x=1\\y=2}} = 2(1-\ln 2)$$

所求切線： $y - 2 = 2(1-\ln 2)(x-1)$

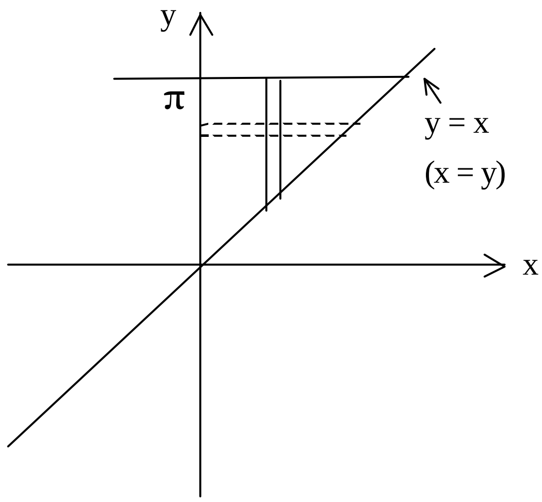
七、

(a) $I=\int_0^{\pi}\int_x^{\pi}\frac{\sin y}{y}dydx$

$=\int_0^{\pi}[\int_0^{y}\frac{\sin y}{y}dx]\,dy$

$=\int_0^{\pi}[\frac{\sin y}{y}x\Big|_0^y]dy=\int_0^{\pi}\sin y dy$

$=(-\cos y)\Big|_0^{\pi}=[-(-1)]-[-1]=2$



(b) $I=\frac{1}{\pi}\iint_{R^2}\frac{1}{(1+x^2+y^2)^{2001}}dxdy$

令 $x=\gamma\cos\theta$, $y=\gamma\sin\theta$

則 $I=\frac{1}{\pi}\int_0^{2\pi}[\int_0^{\infty}\frac{1}{(1+\gamma^2)^{2001}}\cdot\gamma d\gamma]\,d\theta$

$=\frac{1}{\pi}[\int_0^{2\pi}d\theta]\cdot[\int_0^{\infty}\frac{1}{(1+\gamma^2)^{2001}}\cdot\gamma d\gamma]$

$=\frac{1}{\pi}\cdot 2\pi\cdot[(\frac{1}{2})(-\frac{1}{2000})\frac{1}{(1+\gamma^2)^{2000}}]\Big|_0^{\infty}$

$=\frac{1}{\pi}\cdot 2\pi\cdot[0-(-\frac{1}{4000})]$

$=\frac{1}{2000}$

八、

(a) $I=\int_{\Gamma}(2x^2+4xy)dx+(2x^2-y^2)dy$

令 $M(x,y)=2x^2+4xy,\ N(x,y)=2x^2-y^2$

$\frac{\partial M}{\partial x}=4x+4y\ \frac{\partial M}{\partial y}=4x\ ,\ \frac{\partial N}{\partial x}=4x\ ,\ \frac{\partial N}{\partial y}=-2y$

$\because M(x,y)\ ,\ N(x,y)\ ,\ \frac{\partial M}{\partial x},\frac{\partial M}{\partial y},\frac{\partial N}{\partial x},\frac{\partial N}{\partial y}\ \varepsilon$ cont on R^2

且 $\frac{\partial N}{\partial x}=\frac{\partial M}{\partial y}(=4x)$

$\therefore \int_{\Gamma}(2x^2+4xy)dx+(2x^2-y^2)dy\ \varepsilon$ indep

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(b) **由**(a)**之結論，得知：存在** F (x,y)**，使得**

$\frac{\partial F}{\partial x}=M(x,y)\ \frac{\partial F}{\partial y}=N(x,y)$

且 $\int_{(0,0)}^{(1,1)}(2x^2+4xy)dx+(2x^2-y^2)dy=F(x,y)\Big|_{(0,0)}^{(1,1)}$

F (x,y) **求法如下：**

$$\begin{cases}\frac{\partial F}{\partial x}=2x^2+4xy\cdots\cdots(1)\\ \frac{\partial F}{\partial y}=2x^2-y^2\cdots\cdots(2)\end{cases}$$

(1)**得** $F(x,y)=\frac{2}{3}x^2+2x^2y+g(y)\cdots\cdots(3)$

(3)**代入**(2) $2x^2+g'(y)=2x^2-y^2\Rightarrow g'(y)=-y^2$

$\Rightarrow g(y)=\frac{-1}{3}y^3+c\ ,\ \therefore F(x,y)=\frac{2}{3}x^3+2x^2y-\frac{1}{3}y^3+c$

$\therefore \int_{(0,0)}^{(1,1)}(2x^2+4xy)dx+(2x^2-y^2)dy=\frac{2}{3}x^3+2x^2y-\frac{1}{3}y^3\Big|_{(0,0)}^{(1,1)}$

$=\frac{2}{3}+2-\frac{1}{3}=\frac{7}{3}$

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理由： $f(x) \in \text{diff at } x=1 \Rightarrow f'(1)\text{exist}$

$$\Rightarrow f'_{-}(1) = f'_{+}(1) \Rightarrow \lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}$$

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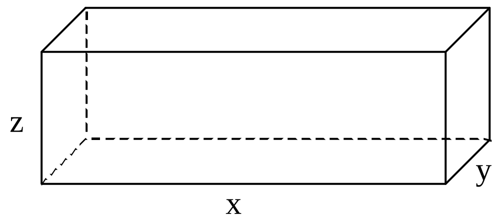

其中 $xyz = 500$

並令 $g(x,y,z) = xyz - 500$

$$\begin{aligned}\omega &= f(x,y,z) + \lambda g(x,y,z)\\ &= xy + 2xz + 2yz + \lambda(xyz - 500)\end{aligned}$$

根據 Lagrornge' s multiplier method.

$F(x,y,z)$ **呈極值之必要條件：**

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令 (x, y) = (1, 2) **代入上式中**

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$$\Rightarrow 2 = 2\ln 2+\frac{dy}{dx}$$

$$\therefore \left.\frac{dy}{dx}\right|_{\substack{x=1\\y=2}} = 2(1-\ln 2)$$

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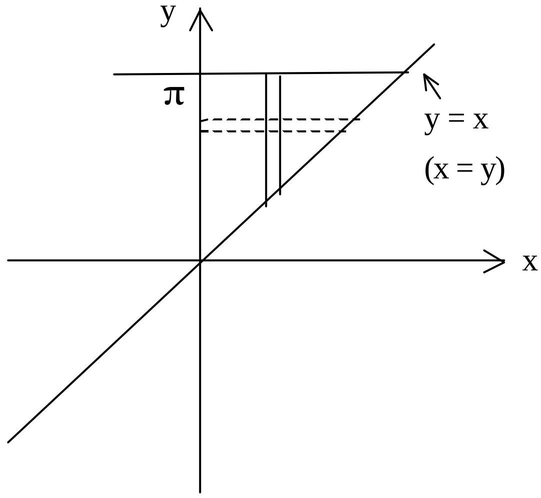
(a) $I = \int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y} dydx$

$= \int_0^{\pi} [\int_0^{y} \frac{\sin y}{y} dx]\, dy$

$= \int_0^{\pi} [\frac{\sin y}{y} x\Big|_0^y]dy = \int_0^{\pi} \sin ydy$

$= (-\cos y)\Big|_0^{\pi} = [-(-1)] - [-1] = 2$



(b) $I = \frac{1}{\pi}\iint_{R^2} \frac{1}{(1+x^2+y^2)^{2001}} dxdy$

令 $x = \gamma\cos\theta$，$y = \gamma\sin\theta$

則 $I = \frac{1}{\pi}\int_0^{2\pi} [\int_0^{\infty} \frac{1}{(1+\gamma^2)^{2001}} \cdot \gamma d\gamma\ ]\, d\theta$

$= \frac{1}{\pi}[\int_0^{2\pi} d\theta] \cdot [\int_0^{\infty} \frac{1}{(1+\gamma^2)^{2001}} \cdot \gamma d\gamma\ ]$

$= \frac{1}{\pi} \cdot 2\pi \cdot [(\frac{1}{2})(-\frac{1}{2000})\frac{1}{(1+\gamma^2)^{2000}}]\Big|_0^{\infty}$

$= \frac{1}{\pi} \cdot 2\pi \cdot [0 - (-\frac{1}{4000})]$

$= \frac{1}{2000}$

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(a) $I = \int_{\Gamma}(2x^2 + 4xy)dx + (2x^2 - y^2)dy$

令 $M(x, y) = 2x^2 + 4xy,\ N(x, y) = 2x^2 - y^2$

$\frac{\partial M}{\partial x} = 4x + 4y\ \frac{\partial M}{\partial y} = 4x\ ,\ \frac{\partial N}{\partial x} = 4x\ ,\ \frac{\partial N}{\partial y} = -2y$

$\because M(x, y)\ ,\ N(x, y)\ ,\ \frac{\partial M}{\partial x}, \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}, \frac{\partial N}{\partial y}\ \varepsilon$ cont on R^2

且 $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}(= 4x)$

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且 $\int_{(0,0)}^{(1,1)}(2x^2 + 4xy)dx + (2x^2 - y^2)dy = F(x, y)\Big|_{(0,0)}^{(1,1)}$

F (x,y) **求法如下：**

$$\begin{cases} \frac{\partial F}{\partial x} = 2x^2 + 4xy \cdots\cdots(1) \\ \frac{\partial F}{\partial y} = 2x^2 - y^2 \cdots\cdots(2) \end{cases}$$

(1)**得** $F(x, y) = \frac{2}{3}x^2 + 2x^2y + g(y) \cdots\cdots(3)$

(3)**代入**(2) $2x^2 + g'(y) = 2x^2 - y^2 \Rightarrow g'(y) = -y^2$

$$\Rightarrow g(y) = \frac{-1}{3}y^3 + c\ ,\ \therefore F(x, y) = \frac{2}{3}x^3 + 2x^2y - \frac{1}{3}y^3 + c$$

$$\therefore \int_{(0,0)}^{(1,1)}(2x^2 + 4xy)dx + (2x^2 - y^2)dy = \frac{2}{3}x^3 + 2x^2y - \frac{1}{3}y^3\Big|_{(0,0)}^{(1,1)}$$

$$= \frac{2}{3} + 2 - \frac{1}{3} = \frac{7}{3}$$